

# Hardness of approximation for orthogonal rectangle packing and covering problems

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## Abstract

Bansal and Sviridenko [4] proved that there is no asymptotic PTAS for 2-DIMENSIONAL ORTHOGONAL BIN PACKING (without rotations), unless  $P = NP$ . We show that similar approximation hardness results hold for several 2- and 3-dimensional rectangle packing and covering problems even if rotations by ninety degrees are allowed. Moreover, for some of these problems we provide explicit lower bounds on asymptotic approximation ratio of any polynomial time approximation algorithm. Our hardness results apply to the most studied case of 2-dimensional problems with unit square bins, and for 3-dimensional strip packing and covering problems with a strip of unit square base.

## 1 Introduction

Bin packing and covering problems have many real-world applications in areas like job scheduling, container loading, and cutting objects out of a strip of material in such a way that the amount of material wasted is minimal. In this paper we present approximation hardness results for 2-dimensional orthogonal rectangle bin packing and covering problems with unit square bins, and for 3-dimensional strip packing and covering problems. The hardness results are obtained not only for problems with orientation of rectangles fixed, but also for their variants with ninety-degree rotations allowed. Throughout this paper we only consider offline versions of the problems.

**Notation and terminology.** For convenience, the terminology that has been introduced in the study of bin packing problems will be used in this paper for covering problems as well. In all 2-dimensional problems studied below, the input consists of a list  $\mathcal{L} = \{R^1, R^2, \dots, R^n\}$  of 2-dimensional rectangles in the Euclidean space  $\mathbb{R}^2$  and a 2-dimensional rectangular bin  $\mathbb{B} = [0, b_1] \times [0, b_2]$  (for which the notation  $(b_1, b_2)$  is used as well). Each rectangle  $R^i$  is given with an (initial) *orientation* with respect to the coordinate axes, and with side-lengths  $(w(R^i), h(R^i))$  called width and height, respectively. The generalization to the higher dimensions is straightforward. In packing problems rectangles of  $\mathcal{L}$  have to be packed into a bin without overlapping, in covering problems rectangles can overlap and be placed (partially) outside of a bin. The most interesting and well-studied version of these problems is the so-called *orthogonal* version, where the edges of placed rectangles and a bin have to be parallel to the coordinate axes. In packing and covering problems *without rotations* rectangles have to be placed with a given (initial) orientation and a feasible solution is called *oriented packing* and *oriented covering*, respectively. In problems *with*

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rotations allowed rectangles to be placed may be rotated around any of the axes by ninety degrees and a feasible solution is referred to as  $r$ -packing and  $r$ -covering, respectively.

**Definition 1** Given a list  $\mathcal{L}$  of 2-dimensional rectangles and a 2-dimensional bin  $\mathbb{B} = (b_1, b_2)$ . The goal of the problems 2-DIMENSIONAL BIN PACKING (2-BP) and 2-DIMENSIONAL BIN PACKING WITH ROTATIONS (2-BP<sup>r</sup>) is to find an oriented packing and an  $r$ -packing of all rectangles of  $\mathcal{L}$  into the minimum number of copies of a bin  $\mathbb{B}$ , respectively.

In the problems 2-DIMENSIONAL BIN COVERING (2-BC) and 2-DIMENSIONAL BIN COVERING WITH ROTATIONS (2-BC<sup>r</sup>) one is looking for an oriented covering and an  $r$ -covering by rectangles from  $\mathcal{L}$ , that maximize the number of completely covered copies of a bin  $\mathbb{B}$ , respectively.

**Definition 2** In 3-dimensional strip versions of the problems a list  $\mathcal{L}$  of 3-dimensional rectangles and a 3-dimensional bin  $\mathbb{B} = (b_1, b_2, \infty)$  with the unlimited third side-length (a strip) are given. In the problems 3-DIMENSIONAL STRIP PACKING (3-SP) and 3-DIMENSIONAL STRIP PACKING WITH ROTATIONS (3-SP<sup>r</sup>) one has to find an oriented packing and an  $r$ -packing that minimizes  $h$  such that all rectangles of  $\mathcal{L}$  are packed into the bin  $(b_1, b_2, h)$ , respectively. If only ninety degree rotations around the  $z$ -axis (the unlimited direction of the strip  $\mathbb{B}$ ) are allowed, the problem is called  $z$ -ORIENTED 3-DIMENSIONAL STRIP PACKING.

The goal of the 3-DIMENSIONAL STRIP COVERING problem (3-SC) is to maximize  $h$  such that the part  $(b_1, b_2, h)$  of the strip  $\mathbb{B}$  is completely covered. We also consider the  $z$ -ORIENTED 3-DIMENSIONAL STRIP COVERING problem (3-SC<sup>z</sup>), where rectangles can be rotated by ninety degrees around the  $z$ -axis.

In many bin packing problems, approximation hardness results are already achieved on instances requiring only a very small number of bins (like one, two, or three). To describe better the real difficulty of these problems, the *asymptotic approximation ratio* has become the standard measure used to analyse the quality of approximation algorithms. For an approximation algorithm  $\mathcal{A}$  solving the minimization problem the *asymptotic approximation ratio*  $\rho_{\mathcal{A}}^{\infty}$  is defined as

$$\rho_{\mathcal{A}}^{\infty} = \lim_{n \rightarrow \infty} \sup_I \left\{ \frac{\mathcal{A}(I)}{\text{OPT}(I)} : \text{OPT}(I) \geq n \right\},$$

where  $I$  ranges over the set of all problem instances, and  $\mathcal{A}(I)$  (resp.  $\text{OPT}(I)$ ) denote the value of the solution returned by  $\mathcal{A}$  (resp. the optimum value) for an input instance  $I$ . For a maximization problem,  $\frac{\mathcal{A}(I)}{\text{OPT}(I)}$  is replaced by  $\frac{\text{OPT}(I)}{\mathcal{A}(I)}$  so that always  $\rho_{\mathcal{A}}^{\infty} \geq 1$ . We say that a problem admits an *asymptotic polynomial time approximation scheme* (shortly, APTAS), if for any  $\varepsilon > 0$  there is a polynomial time algorithm with an asymptotic approximation ratio less than  $1 + \varepsilon$ . If, furthermore, there is an algorithm whose running time is polynomial in  $|I|$  and  $1/\varepsilon$ , the problem admits an *asymptotic fully polynomial time approximation scheme*. For other optimization terminology we refer to Ausiello et al. [1].

**Overview.** As bin packing and covering problems are known to be NP-hard, the research has concentrated on the design of polynomial time approximation algorithms and schemes for them. The following is a brief overview of known results.

For 1-BP, Fernandez de la Vega & Lueker [11] designed an APTAS. More precisely, for any positive integer  $k$  they provided a polynomial time algorithm that uses at most  $(1 + \frac{1}{k})\text{OPT} + 1$  bins. Later, Karmarkar & Karp [18] gave for this problem a single algorithm with asymptotic approximation ratio 1 that uses  $\text{OPT} + O(1 + \log^2 \text{OPT})$  bins. For the 2-BP problem Caprara [5] presented an algorithm with currently the best asymptotic approximation ratio 1.691. Bansal

& Sviridenko [4] provided an APTAS for a restricted version of  $d$ -DIMENSIONAL BIN PACKING in which rectangles and bins are  $d$ -cubes; this result was independently obtained by Correa & Kenyon [8]. For 3-BP, Li & Cheng [20] and Csizik & van Vliet [10] designed algorithms with asymptotic approximation ratio at most 4.84. This ratio was improved to  $4 + \varepsilon$  by Jansen & Solis-Oba [14]. The algorithms from [20] and [10] work also in the  $d$ -dimensional case of BIN PACKING with an asymptotic approximation ratio at most  $1.691^d$ . For the problem 2-SP, the breakthrough result was obtained by Kenyon & Rémila [19] who gave an APTAS. For 3-SP, Miyazawa & Wakabayashi [22] presented an algorithm with asymptotic approximation ratio at most 2.64, which was improved to  $2 + \varepsilon$  by Jansen & Solis-Oba [14].

When ninety-degree rotations are allowed, only weaker results are known. Some algorithms for the versions without rotations provide upper bounds on asymptotic approximation ratio for versions with rotations as well. The results by Miyazawa & Wakabayashi [21] were the first ones where rotations are exploited in a non-trivial way. Currently the best upper bounds on asymptotic approximation ratio for the problems 2-BP<sup>r</sup>, 3-BP<sup>r</sup>, 3-SP<sup>r</sup>, and 3-SP<sup>z</sup>, are  $2 + \varepsilon$ , 4.89, 2.76, and 2.64, respectively, see [22] and [15]. Moreover, Jansen & Stee designed an APTAS for 2-SP<sup>r</sup> ([15]).

Though bin covering problems are considered to be dual to the bin packing problems, many techniques used for bin packing problems do not seem to be adaptable for bin covering problems. In spite of many improvements for bin packing problems, only a few results for bin covering problems are known. Let us mention among them an APTAS for 1-BC by Csirik et al. ([9]) improved later to an *asymptotic fully polynomial time approximation scheme* by Jansen & Solis-Oba ([13]).

In spite of a great deal of efforts the questions about the existence of an APTAS have been open for a long time for several basic bin packing and covering problems. The research around the PCP-Theorem paved the way to the negative answers to these questions. For some problems it has become known that even to approximate the solution by an algorithm with asymptotic approximation ratio close to 1 is NP-hard. However, due to lack of universal methods of designing NP-hard gap preserving reductions to the packing and covering problems, there is only a few results in this area. Unless  $P = NP$ , non-existence of APTAS was proved for the 2-DIMENSIONAL VECTOR BIN COVERING by Woeginger ([24]), and for 2-DIMENSIONAL BIN PACKING by Bansal & Sviridenko ([4]). These approximation hardness results and their proofs apply to some higher dimensional packing and covering problems as well.

**Rectangle Packing and Covering without and with Rotations.** When dealing with packing and covering problems *without rotation*, one can always assume that a bin is a unit square (resp., a base of a strip is a unit square), as the problems are invariant under *heterogeneous scaling*, i.e., the one which scales by different factors in different coordinate directions. However, this is not true for problems with rotations allowed. It is unclear if for the problems with rotations allowed their restricted variants with a unit square bin are easier to approximate than the general one or not. For some problems algorithms with better asymptotic approximation ratio were suggested in such restricted case. For example, when a base of a strip in 3-SP<sup>z</sup> is a unit square, an algorithm with asymptotic approximation ratio at most 2.528 (instead of 2.64 in general case) is known [21].

Using heterogeneous scaling we can show that 2-BP can be viewed as a particular case of general 2-BP<sup>r</sup> with highly excentric instances. Let a list  $\mathcal{L} = \{R^1, R^2, \dots, R^n\}$  of rectangles with dimensions  $R^i = (w(R^i), h(R^i))$ ,  $i = 1, 2, \dots, n$ , and a bin  $\mathbb{B} = (b_1, b_2)$  be an instance of 2-BP. We can find positive scaling factor  $\lambda$  and transform any  $R^i$  of  $\mathcal{L}$  to  $R_\lambda^i = (\lambda w(R^i), h(R^i))$  and the bin  $\mathbb{B}$  to  $\mathbb{B}_\lambda = (\lambda b_1, b_2)$ , so that the minimum  $\min\{\lambda w(R^i) : 1 \leq i \leq n\} > b_2$ . It is easy to see that even if ninety-degree rotations are allowed, the only way how a rescaled rectangle  $R_\lambda^i$  can fit in the rescaled bin  $\mathbb{B}_\lambda$  is for  $R_\lambda^i$  being in the initial orientation. Similarly, 3-SP can be handled as a particular case of 3-SP<sup>r</sup> or 3-SP<sup>z</sup>. In such a way a heterogeneous scaling can be used to reduce

oriented packing problems to the ones with ninety-degree rotations allowed. Thus, for problems 2-BP<sup>r</sup>, 3-SP<sup>r</sup>, and 3-SP<sup>z</sup> without any à priori restrictions on the shape of a bin, non-existence of an APTAS follows from results by Bansal & Sviridenko ([4], see also [3]) for 2-BP. However, for the most interesting case of 2-BP<sup>r</sup> with a unit *square* bin, one can hardly obtain similar approximation hardness results directly from those mentioned above.

To the best knowledge of authors, no similar approximation hardness results were known prior this work for rectangle packing problems with rotations allowed in case of a unit square bin, and for rectangle covering problems at all.

**Main results.** In this paper we prove non-existence of an APTAS (unless  $P = NP$ ) for 2-DIMENSIONAL BIN PACKING WITH ROTATIONS into unit square bins (Section 3). The methods allow to provide an explicit lower bound on asymptotic approximation ratio of any polynomial time approximation algorithm (unless  $P = NP$ ). For example, we provide a lower bound  $1 + \frac{1}{3792}$  for 2-DIMENSIONAL BIN PACKING WITH ROTATIONS, and  $1 + \frac{1}{2196}$  for the same problem without rotations. In Section 4 we develop methods suitable for covering counterparts of packing problems, and prove similar approximation hardness results for 2-DIMENSIONAL BIN COVERING and 2-DIMENSIONAL BIN COVERING WITH ROTATIONS using unit square bins.

In Section 5 we apply the above results and derive non-existence of an APTAS (unless  $P = NP$ ) for the problems 3-DIMENSIONAL STRIP PACKING WITH ROTATIONS,  $z$ -ORIENTED 3-DIMENSIONAL STRIP PACKING, 3-DIMENSIONAL STRIP COVERING, and  $z$ -ORIENTED 3-DIMENSIONAL STRIP COVERING; all these hardness results apply to the case of strip with unit square base.

We prove also non-existence of an APTAS for a related problem in which the goal is to pack maximum number of 3-dimensional rectangles from a given collection into a *single* cube bin (Section 6).

## 2 Gap preserving reductions from Max-3DM

Approximation hardness results for bounded MAXIMUM 3-DIMENSIONAL MATCHING suit well as a starting point to inapproximability results for various (multidimensional) packing, covering, and scheduling problems, see e.g., [24], [6], and [4]. In this section we demonstrate this approach and present general gap preserving reductions (using various parameters) from a bounded MAX-3DM to bin packing and covering problems.

**Definition 3** *Given three pairwise disjoint sets  $X$ ,  $Y$ , and  $Z$ , and a set of ordered triples  $T \subseteq X \times Y \times Z$ . Without loss of generality we assume that any element of  $X \cup Y \cup Z$  occurs in at least one triple in  $T$ . A matching in  $T$  is a subset  $M \subseteq T$  such that no two ordered triples in  $M$  agree in any coordinate. The goal of the MAXIMUM 3-DIMENSIONAL MATCHING problem (shortly, MAX-3DM) is to find a matching in  $T$  of maximum cardinality. The  $k$ -BOUNDED MAX-3DM problem is restricted to instances of MAX-3DM such that any element in  $X$ ,  $Y$ , and  $Z$  occurs in at most  $k$  triples in  $T$ .*

Kann [17] showed that the 3-BOUNDED MAX-3DM problem is Max SNP-complete (hence also APX-complete). Thus, using the PCP-theorem, the existence of a PTAS for it would imply that  $P = NP$ . Petrank [23] proved a refined approximation hardness result that an NP-hard gap occurs also on instances with perfect matching. For some purposes, for example, to achieve explicit inapproximability results, it is more convenient to use the following NP-hard gap type result [7] valid for instances MAX-3DM with the property that any element in  $X$ ,  $Y$ , and  $Z$  occurs in exactly 2 triples in  $T$ .

**Theorem A.** [7] *There are instances  $T \subseteq X \times Y \times Z$  of 2-bounded MAX-3DM with  $|X| = |Y| = |Z| (= q)$  and every element of  $X \cup Y \cup Z$  occurring in exactly 2 triples in  $T$  such that it is NP-hard to distinguish between instances with  $\text{OPT}(T) > 0.979338843q$  and  $\text{OPT}(T) < 0.9690082645q$ .*

Now we build on ideas of the gap preserving reduction introduced by Bansal & Sviridenko [4] in their proof of non-existence of an APTAS for 2-DIMENSIONAL BIN PACKING. We show that similar reductions from MAX-3DM can be used to prove the same approximation hardness results for 2-dimensional bin packing and covering problems with unit square bins and with rotations by ninety degrees allowed.

**The Bin Packing reduction.** Let  $\mathcal{T}$  be an infinite set of instances (ordered triples)  $T$  of MAX-3DM with the optimum value  $\text{OPT}(T)$  and with the property that for some efficiently computable functions  $\alpha(T)$ ,  $\beta(T)$  it is NP-hard to decide of whether  $\text{OPT}(T) \geq \beta(T)$ , or  $\text{OPT}(T) < \alpha(T)$ . (Notice, that Theorem A describes a particular NP-hard gap result of this type.) For a fixed instance  $T \in \mathcal{T}$  we define the (pairwise disjoint) sets  $X$ ,  $Y$ ,  $Z$  as the projections of  $T$  to the first, the second, and the third coordinate, respectively. The elements in  $X$ ,  $Y$ ,  $Z$ , and  $T$  will be denoted as  $\{x_i : 1 \leq i \leq |X|\}$ ,  $\{y_j : 1 \leq j \leq |Y|\}$ ,  $\{z_k : 1 \leq k \leq |Z|\}$ , and  $\{t_l : 1 \leq l \leq |T|\}$ , respectively. Of course, any  $t_l \in T$  is of the form  $t_l = (x_i, y_j, z_k) \in X \times Y \times Z$ . Let  $n = |X| + |Y| + |Z|$ ,  $q = \max\{|X|, |Y|, |Z|\}$ , and  $r = 32q$ . (In fact, we will use this reduction for instances from Theorem A, where  $|X| = |Y| = |Z| = \frac{1}{2}|T|$  holds.) The parameters of the reduction are a gap location  $\beta(T)$  and a constant  $\delta \in (0, 10^{-4}]$ .

We start with the definition of an integer for each element in  $X$ ,  $Y$ ,  $Z$ , and  $T$  as follows:  $x'_i = ir^3 + i^2r + 1$ , for  $1 \leq i \leq |X|$ ,  $y'_j = jr^6 + j^2r^4 + 2$ , for  $1 \leq j \leq |Y|$ ,  $z'_k = kr^9 + k^2r^7 + 4$ , for  $1 \leq k \leq |Z|$ . For each triple  $t_l = (x_i, y_j, z_k) \in T$  we define an integer  $t'_l = r^{10} - x'_i - y'_j - z'_k + 15$ . Put  $c = \frac{r^{10}+15}{\delta}$  and observe that  $0 < x'_i, y'_j, z'_k < \frac{\delta c}{10}$  for all  $i, j, k$ , and  $t'_l + x'_i + y'_j + z'_k = c\delta$  whenever  $t_l = (x_i, y_j, z_k) \in T$ .

In what follows we describe a collection of rectangles  $\mathcal{R}_T$  used in the Bin Packing reduction. For each  $x_i \in X$  (resp.,  $y_j \in Y$  and  $z_k \in Z$ ) we define a pair of rectangles  $A_{X,i}$ ,  $A'_{X,i}$  (resp.,  $A_{Y,j}$ ,  $A'_{Y,j}$  and  $A_{Z,k}$ ,  $A'_{Z,k}$ ) as follows:

$$\begin{aligned} A_{X,i} &= \left(\frac{1}{4} - 4\delta + \frac{x'_i}{c}, \frac{3}{4} + 13\delta - \frac{x'_i}{c}\right) \text{ and } A'_{X,i} = \left(\frac{1}{4} + 4\delta - \frac{x'_i}{c}, \frac{1}{4} - 13\delta + \frac{x'_i}{c}\right), \\ A_{Y,j} &= \left(\frac{1}{4} - 3\delta + \frac{y'_j}{c}, \frac{3}{4} + 12\delta - \frac{y'_j}{c}\right) \text{ and } A'_{Y,j} = \left(\frac{1}{4} + 3\delta - \frac{y'_j}{c}, \frac{1}{4} - 12\delta + \frac{y'_j}{c}\right), \\ A_{Z,k} &= \left(\frac{1}{4} - 2\delta + \frac{z'_k}{c}, \frac{3}{4} + 11\delta - \frac{z'_k}{c}\right) \text{ and } A'_{Z,k} = \left(\frac{1}{4} + 2\delta - \frac{z'_k}{c}, \frac{1}{4} - 11\delta + \frac{z'_k}{c}\right). \end{aligned}$$

For each  $t_l \in T$  we define two rectangles  $B_l$  and  $B'_l$  such that

$$B_l = \left(\frac{1}{4} + 8\delta + \frac{t'_l}{c}, \frac{3}{4} + 10\delta - \frac{t'_l}{c}\right) \text{ and } B'_l = \left(\frac{1}{4} - 8\delta - \frac{t'_l}{c}, \frac{1}{4} - 10\delta + \frac{t'_l}{c}\right).$$

Let  $\mathcal{D}$  be a collection of  $|T| + n - 4\beta(T)$  dummy rectangles of the same size  $(\frac{3}{4} - 10\delta, 1)$ .

Let  $\mathcal{A}_X = \{A_{X,1}, A_{X,2}, \dots, A_{X,|X|}\}$ ,  $\mathcal{A}'_X = \{A'_{X,1}, A'_{X,2}, \dots, A'_{X,|X|}\}$  and define sets of rectangles  $\mathcal{A}_Y$ ,  $\mathcal{A}'_Y$ ,  $\mathcal{A}_Z$ , and  $\mathcal{A}'_Z$  analogously. Put  $\mathcal{A} = \mathcal{A}_X \cup \mathcal{A}_Y \cup \mathcal{A}_Z$  and  $\mathcal{A}' = \mathcal{A}'_X \cup \mathcal{A}'_Y \cup \mathcal{A}'_Z$ . Similarly, let  $\mathcal{B} = \{B_1, B_2, \dots, B_{|T|}\}$  and  $\mathcal{B}' = \{B'_1, B'_2, \dots, B'_{|T|}\}$ .

**Remark.** The sizes of rectangles are closely related to those used in [4]. This allows us to avoid repeating proofs of some of their properties; we can simply refer to [4]. To explain how our

reduction is related to their, let us assume that heights of rectangles from  $\mathcal{A} \cup \mathcal{B}$  (respectively,  $\mathcal{A}' \cup \mathcal{B}'$ ) are increased by  $p - \frac{1}{4} - 9\delta$  (respectively,  $\frac{1}{4} + 9\delta - p$ ), where  $p$  is a new parameter.

The reduction introduced by Bansal & Sviridenko ([4]) can be viewed as a particular case of such parametrized Bin Packing reduction with  $p = 0$ ,  $\delta$  arbitrarily small, a set  $\mathcal{T}$  of instances  $T \subseteq X \times Y \times Z$  of 3-bounded MAX-3DM with  $|X| = |Y| = |Z| = q$ , and a gap location  $\beta(T) = q$  guaranteed by the Petrank's result.

Any choice of  $p \in [\frac{1}{4} + 9\delta, \frac{1}{2} - 20\delta]$  allows us to prove some additional properties that are crucial for the Bin Packing problem with rotations allowed. The choice of  $p = \frac{1}{4} + 9\delta$  provides the best explicit approximation lower bounds for this problem.

**The Bin Covering reduction.** The Bin Covering reduction is very similar to the Bin Packing reduction, the only difference are heights of rectangles. For each  $x_i \in X$  (resp.,  $y_j \in Y$  and  $z_k \in Z$ ) we define a pair of rectangles  $A_{X,i}$ ,  $A'_{X,i}$  (resp.,  $A_{Y,j}$ ,  $A'_{Y,j}$  and  $A_{Z,k}$ ,  $A'_{Z,k}$ ) as follows:

$$\begin{aligned} A_{X,i} &= \left( \frac{1}{4} - 4\delta + \frac{x'_i}{c}, \frac{5}{28} - \delta - \frac{x'_i}{c} \right) \text{ and } A'_{X,i} = \left( \frac{1}{4} + 4\delta - \frac{x'_i}{c}, \frac{23}{28} + \delta + \frac{x'_i}{c} \right), \\ A_{Y,j} &= \left( \frac{1}{4} - 3\delta + \frac{y'_j}{c}, \frac{5}{28} - 2\delta - \frac{y'_j}{c} \right) \text{ and } A'_{Y,j} = \left( \frac{1}{4} + 3\delta - \frac{y'_j}{c}, \frac{23}{28} + 2\delta + \frac{y'_j}{c} \right), \\ A_{Z,k} &= \left( \frac{1}{4} - 2\delta + \frac{z'_k}{c}, \frac{5}{28} - 3\delta - \frac{z'_k}{c} \right) \text{ and } A'_{Z,k} = \left( \frac{1}{4} + 2\delta - \frac{z'_k}{c}, \frac{23}{28} + 3\delta + \frac{z'_k}{c} \right). \end{aligned}$$

For each  $t_l \in T$  we define two rectangles  $B_l$  and  $B'_l$  such that

$$B_l = \left( \frac{1}{4} + 8\delta + \frac{t'_l}{c}, \frac{5}{28} - \frac{t'_l}{c} \right) \quad \text{and} \quad B'_l = \left( \frac{1}{4} - 8\delta - \frac{t'_l}{c}, \frac{23}{28} + \frac{t'_l}{c} \right).$$

Let  $\mathcal{D}$  be a collection of  $|T| + n - 4\beta(T)$  dummy rectangles of the same size  $(\frac{3}{4} + 9\delta, 1)$  and define sets  $\mathcal{A}$ ,  $\mathcal{A}'$ ,  $\mathcal{B}$ , and  $\mathcal{B}'$  in the same way as in the Bin Packing reduction.

The collection of rectangles  $\mathcal{R}_T = \mathcal{A} \cup \mathcal{A}' \cup \mathcal{B} \cup \mathcal{B}' \cup \mathcal{D}$ , where  $\mathcal{A}$ ,  $\mathcal{A}'$ ,  $\mathcal{B}$ ,  $\mathcal{B}'$ , and  $\mathcal{D}$  are corresponding sets for Bin Packing (resp. Bin Covering) reduction along with a unit square bin is now viewed as an instance of the 2-BP<sup>r</sup> (resp. 2-BC and 2-BC<sup>r</sup>). To show that NP-hard gap of MAX-3DM is preserved for these problems, we relate in Sections 3 and 4 the optimum value of 2-BP<sup>r</sup> (resp. 2-BC, and 2-BC<sup>r</sup>) for an instance  $\mathcal{R}_T$  to the optimum  $\text{OPT}(T)$  of MAX-3DM for an instance  $T$ .

First we observe some basic properties of rectangles from the collection  $\mathcal{R}_T$ . As the side-lengths of a pair of rectangles defined for the same element from  $X$ ,  $Y$ ,  $Z$ , and  $T$  have similar properties in the both reductions above and also in Bansal & Sviridenko's reduction, some of results from [4] are preserved to our case. We start with the concept of buddies introduced in [4] for a pair of rectangles, and recall their important properties.

**Definition 4** *We say that two rectangles  $A$  and  $A'$  from  $\mathcal{A} \cup \mathcal{A}' \cup \mathcal{B} \cup \mathcal{B}'$  are buddies if  $\{A, A'\}$  correspond to a pair of rectangles for a single element from  $X$ ,  $Y$ ,  $Z$  or  $T$ , e.g.,  $\{A, A'\} = \{A_{X,i}, A'_{X,i}\}$  for some  $x_i \in X$ .*

**Observation 1** *For any two rectangles  $A$ ,  $A'$  in  $\mathcal{A} \cup \mathcal{A}' \cup \mathcal{B} \cup \mathcal{B}'$ ,  $h(A) + h(A') = 1$  if and only if  $A$  and  $A'$  are buddies.*

The proofs of the following Lemmas 1 and 2 can be found in [4] and work in our setting as well, as widths of rectangles are the same in all reductions.

**Lemma 1** For any rectangles  $A_1, A_2, A_3 \in \mathcal{A}$  and  $B \in \mathcal{B}$ ,  $w(A_1) + w(A_2) + w(A_3) + w(B) = 1$  if and only if  $\{A_1, A_2, A_3, B\} = \{A_{X,i}, A_{Y,j}, A_{Z,k}, B_l\}$  for some integers  $i, j, k$ , and  $l$  such that  $t_l = (x_i, y_j, z_k) \in T$ . A similar statement holds also for rectangles  $A'_1, A'_2, A'_3 \in \mathcal{A}'$ ,  $B' \in \mathcal{B}'$ .

**Lemma 2** Let  $A_1, A_2, A_3, A_4 \in \mathcal{A} \cup \mathcal{A}'$  be such that no two of them are buddies. Then  $\sum_{i=1}^4 w(A_i) \neq 1$ .

### 3 Two-Dimensional Bin Packing problem with Rotations

In this section we prove non-existence of APTAS for 2-BP<sup>r</sup> with unit square bins. We show that for any  $T \in \mathcal{T}$  the side-lengths in the Bin Packing reduction are chosen such that if  $\text{OPT}(T) \geq \beta(T)$ , the rectangles can be packed into bins such that their number is at most  $|T| + n - 3\beta(T)$ . On the other hand, if  $\text{OPT}(T) < \alpha(T)$ , then the number of bins needed to pack all rectangles from  $\mathcal{R}_T$  is larger than  $|T| + n - 3\beta(T)$  by a constant multiplicative factor larger than 1.

**Observation 2** For any rectangle from the Bin Packing reduction,  $A \in \mathcal{A}$  implies  $w(A) + h(A) = 1 + 9\delta$ ,  $A' \in \mathcal{A}'$  implies  $w(A') + h(A') = \frac{1}{2} - 9\delta$ ,  $B \in \mathcal{B}$  implies  $w(B) + h(B) = 1 + 18\delta$ , and  $B' \in \mathcal{B}'$  implies  $w(B') + h(B') = \frac{1}{2} - 18\delta$ .

**Lemma 3** (i) In an  $r$ -packing of a unit square bin the rectangles from  $\mathcal{A} \cup \mathcal{B}$  are either all in the initial orientation, or all are rotated by ninety degrees.

(ii) If an  $r$ -packing of a unit square bin contains exactly four rectangles from  $\mathcal{A} \cup \mathcal{B}$ , then necessarily the rectangles from  $\mathcal{A} \cup \mathcal{B} \cup \mathcal{A}' \cup \mathcal{B}'$  packed in this bin are either all in the initial orientation, or all are rotated by ninety degrees.

*Proof.* (i) It easily follows from the fact that any rectangle from  $\mathcal{A} \cup \mathcal{B}$  has width at least  $\frac{1}{4} - 4\delta$  and height at least  $\frac{3}{4} + 9\delta$ .

(ii) By part (i), all four rectangles from  $\mathcal{A} \cup \mathcal{B}$  contained in the bin are either simultaneously in the initial orientation, or all are rotated by ninety degrees. We can assume that they are in the initial orientation, the discussion in the latter case is the same. As height of each of them is  $> \frac{3}{4} + 9\delta$ , any line in  $y$ -direction (i.e., parallel to  $y$ -axis) intersects the interior of at most one rectangle from  $\mathcal{A} \cup \mathcal{B}$ . Moreover, the sum of widths of those four rectangles is  $> 1 - 16\delta$ . Consequently, if another rectangle  $A$  (rotated, or not) is packed in this bin, then some line in  $y$ -direction intersects interiors of both,  $A$  and one some rectangle from  $\mathcal{A} \cup \mathcal{B}$ . It easily follows that  $A$  is in its initial orientation as well, as rotated  $A$  would be too high to fit. Consequently, the rectangles in the bin has to be packed either all in the initial orientation, or all rotated by ninety degrees.  $\square$

**Lemma 4** Let an  $r$ -packing of a unit square bin contain exactly four rectangles from  $\mathcal{A} \cup \mathcal{B}$ . Then at most 8 rectangles from  $\mathcal{A} \cup \mathcal{B} \cup \mathcal{A}' \cup \mathcal{B}'$  can be packed in it. Moreover, if exactly 8 such rectangles are packed in it, then for any  $h \in [4\delta, \frac{1}{4} - 13\delta]$ , each rectangle intersects exactly one of lines  $L_1 = \{(x, y) : y = h\}$  and  $L_2 = \{(x, y) : y = 1 - h\}$ .

*Proof.* Assume that an  $r$ -packing of the unit bin  $\mathbb{B}$  contains exactly four rectangles from  $\mathcal{A} \cup \mathcal{B}$  and some rectangles from  $\mathcal{A}' \cup \mathcal{B}'$ . Due to Lemma 3(ii), those rectangles are either all in the initial orientation, or all are rotated by ninety degrees. We can assume that they are all in the initial orientation; the case when all are rotated by ninety degrees could be discussed similarly.

The projections of rectangles from  $\mathcal{A} \cup \mathcal{B}$  on the  $x$ -axis cannot overlap (rectangles are too high) and hence, less than  $16\delta$  of the length of  $[0, 1]$  can be uncovered by them. As width of each rectangle is roughly  $\frac{1}{4}$ , these projections are only small perturbations of intervals  $[0, \frac{1}{4}]$ ,  $[\frac{1}{4}, \frac{1}{2}]$ ,  $[\frac{1}{2}, \frac{3}{4}]$ , and  $[\frac{3}{4}, 1]$ .

Now consider a rectangle  $A'$  from  $\mathcal{A}' \cup \mathcal{B}'$  packed in  $\mathbb{B}$ . The projection of  $A'$  on the  $x$ -axis has to overlap with at least one rectangle  $A$  from  $\mathcal{A} \cup \mathcal{B}$ . As height of  $A$  is larger than  $\frac{3}{4} + 9\delta$ ,  $A'$  is either completely above the line  $\{(x, y) : y = \frac{3}{4} + 9\delta\}$ , or below the line  $\{(x, y) : y = \frac{1}{4} - 9\delta\}$ . It is also easy to see that no line in  $y$ -direction can intersect three rectangles. Hence, if such line intersects interiors of two distinct rectangles from  $\mathcal{A}' \cup \mathcal{B}'$ , then one is located completely above the line  $\{(x, y) : y = \frac{3}{4} + 9\delta\}$  and another one is below the line  $\{(x, y) : y = \frac{1}{4} - 9\delta\}$ . Moreover, the total overlap of projections in  $y$ -direction of rectangles from  $\mathcal{A}' \cup \mathcal{B}'$  in  $\mathbb{B}$  is less than  $16\delta$ . As width of each rectangle from  $\mathcal{A}' \cup \mathcal{B}'$  is roughly  $\frac{1}{4}$ , there can be at most four rectangles packed in  $\mathbb{B}$ . If there are exactly four such rectangles, their projections on the  $x$ -axis are again small perturbations of intervals  $[0, \frac{1}{4}]$ ,  $[\frac{1}{4}, \frac{1}{2}]$ ,  $[\frac{1}{2}, \frac{3}{4}]$ , and  $[\frac{3}{4}, 1]$ .

Let  $A$  be one of four rectangles from  $\mathcal{A} \cup \mathcal{B}$  and assume now that there are exactly four rectangles from  $\mathcal{A}' \cup \mathcal{B}'$  packed in  $\mathbb{B}$ . Clearly,  $A$  has its projection on the  $x$ -axis overlapping with that of a rectangle from  $\mathcal{A}' \cup \mathcal{B}'$ , say  $A'$ . As height of  $A$  is  $> \frac{3}{4} + 9\delta$  and height of  $A'$  is  $> \frac{1}{4} - 13\delta$ , it easily follows that whenever  $h \in [4\delta, \frac{1}{4} - 13\delta]$ , each of both rectangles intersects exactly one of lines  $L_1 = \{(x, y) : y = h\}$  and  $L_2 = \{(x, y) : y = 1 - h\}$ . The rest follows from the fact that the projection of each rectangle from  $\mathcal{A}' \cup \mathcal{B}'$  on the  $x$ -axis overlaps with the projection of some rectangle from  $\mathcal{A} \cup \mathcal{B}$ .  $\square$

**Definition 5** *Given an  $r$ -packing of a unit square bin by some rectangles  $\mathcal{R}_T$  introduced in the Bin Packing reduction. The bin is called well-packed, if it contains four rectangles from  $\mathcal{A} \cup \mathcal{B}$  and four rectangles from  $\mathcal{A}' \cup \mathcal{B}'$ .*

Now the crucial fact is, that we can characterize well-packed bins for  $r$ -packing in terms of an instance  $T$  of MAX-3DM similarly as it has been done in [4] for oriented packing.

**Lemma 5** *A unit square bin is well-packed if and only if it contains the rectangles  $A_{X,i}$ ,  $A_{Y,j}$ ,  $A_{Z,k}$ ,  $B_l$ ,  $A'_{X,i}$ ,  $A'_{Y,j}$ ,  $A'_{Z,k}$ ,  $B'_l$ , for some  $t_l = (x_i, y_j, z_k) \in T$ .*

*Proof.* The 8-tuple of rectangles corresponding to a triple as above can be packed in a unit square bin  $\mathbb{B}$  even without rotations. Starting from the bottom left corner of the bin  $\mathbb{B}$  and moving to the right, each of rectangles  $A_{X,i}$ ,  $A_{Y,j}$ ,  $A_{Z,k}$ , and  $B_l$  is placed such that it touches the bottom of the bin  $\mathbb{B}$  (see Fig. 1). As  $w(A_{X,i}) + w(A_{Y,j}) + w(A_{Z,k}) + w(B_l) = 1$  (Lemma 1), the rectangles can be packed in this way. The rectangles  $A'_{X,i}$ ,  $A'_{Y,j}$ ,  $A'_{Z,k}$ , and  $B'_l$  can be placed in the remaining gaps starting from the top left corner of the bin  $\mathbb{B}$  and moving towards the right touching the top of the bin. Clearly, it is possible as

$$\begin{aligned} w(A'_{X,i}) + w(A'_{Y,j}) + w(A'_{Z,k}) + w(B'_l) &= 1, \\ h(A_{X,i}) + h(A'_{X,i}) &= h(A_{Y,j}) + h(A'_{Y,j}) = h(A_{Z,k}) + h(A'_{Z,k}) = h(B_l) + h(B'_l) = 1, \\ h(A_{X,i}) &> h(A_{Y,j}) > h(A_{Z,k}) > h(B_l), \\ w(A_{X,i}) < w(A'_{X,i}), w(A_{Y,j}) < w(A'_{Y,j}), &\text{ and } w(A_{Z,k}) < w(A'_{Z,k}). \end{aligned}$$

Now we show that any well-packed bin contains rectangles that correspond to a triple in  $T$ . Fix  $h \in [4\delta, \frac{1}{4} - 13\delta]$  and consider the lines  $L_1 = \{(x, y) : y = h\}$  and  $L_2 = \{(x, y) : y = 1 - h\}$ .



Due to Lemma 4, each rectangle must intersect exactly one of the lines  $L_1$  and  $L_2$ . Moreover, as any rectangle has width larger than  $\frac{1}{5}$ , each of lines  $L_1$  and  $L_2$  intersects exactly four rectangles. Let  $\{A_1, A_2, A_3, A_4\}$  denote the rectangles that intersect  $L_1$  such that  $A_i$  is to the left of  $A_j$  for  $i < j$ . Similarly, let  $\{A_5, A_6, A_7, A_8\}$  denote the rectangles that intersect  $L_2$  in the left to right order. Thus, we have that

$$\sum_{i=1}^4 w(A_i) \leq 1, \quad (1)$$

$$\sum_{i=1}^4 w(A_{i+4}) \leq 1. \quad (2)$$

Observe that for each  $i = 1, 2, 3, 4$  the rectangle  $A_i$  must overlap with  $A_{i+4}$  in the  $x$ -coordinate. Thus, we have that

$$h(A_i) + h(A_{i+4}) \leq 1 \quad \text{for } i = 1, 2, 3, 4. \quad (3)$$

From (3) it follows that, for each  $i = 1, 2, 3, 4$ , at most one of  $A_i, A_{i+4}$  belongs to  $\mathcal{A} \cup \mathcal{B}$ . Consequently, for each  $i = 1, 2, 3, 4$  exactly one of  $A_i, A_{i+4}$  is from  $\mathcal{A} \cup \mathcal{B}$  and another one is from  $\mathcal{A}' \cup \mathcal{B}'$ . Using these facts, we can use the same arguments as in [4]:

- (i) First observe that *at most one from rectangles  $\{A_1, \dots, A_8\}$  belongs to  $\mathcal{B}$* . Indeed, if  $k \geq 2$  of them belong to  $\mathcal{B}$  and  $4 - k$  belong to  $\mathcal{A}$ , then the sum of widths of these rectangles from  $\mathcal{A} \cup \mathcal{B}$  would be  $> 1$ , a contradiction with the fact that any line in  $y$ -direction intersects at most one rectangle from  $\mathcal{A} \cup \mathcal{B}$ .
- (ii) *If no rectangle from  $\{A_1, \dots, A_8\}$  belongs to  $\mathcal{B}$ , then the same is true for  $\mathcal{B}'$* . The height of any rectangle in  $\mathcal{B}'$  is larger than  $\frac{1}{4} - 10\delta$  so such rectangle cannot form a pair  $\{A_i, A_{i+4}\}$  with a rectangle from  $\mathcal{A}$ . Thus, in this case four rectangles belong to  $\mathcal{A}$  and four to  $\mathcal{A}'$ . Using Observation 2 we get  $\sum_{i=1}^8 (w(A_i) + h(A_i)) = 6$ , thus it must be the case that each of (1), (2) and (3) must hold with equality. By Observation 1,  $A_i$  and  $A_{i+4}$  are buddies for each  $i = 1, 2, 3, 4$ . In particular, no two rectangles among  $A_1, A_2, A_3$ , and  $A_4$  are buddies. Now Lemma 2 contradicts with  $\sum_{i=1}^4 w(A_i) = 1$  that has been observed earlier. Thus this case is impossible.

So, necessarily *exactly one of rectangles  $\{A_1, A_2, \dots, A_8\}$  belongs to  $\mathcal{B}$* , say  $B_l$ .

- (iii) As, due to (3), no pair  $\{A_i, A_{i+4}\}$  can contain a rectangle from  $\mathcal{B}'$  and a rectangle from  $\mathcal{A}$ , there can be *at most one rectangle from  $\mathcal{B}'$* . But if there are no rectangles from  $\mathcal{B}'$ , then the sum of widths of all 8 rectangles would be  $> 2$ , a contradiction.

Consequently, there is *exactly one rectangle from  $\mathcal{B}'$ , one from  $\mathcal{B}$ , three from  $\mathcal{A}$ , and three from  $\mathcal{A}'$* . Using Observation 2 we get  $\sum_{i=1}^8 (w(A_i) + h(A_i)) = 6$ , thus each of (1), (2), and (3) holds with equality. In particular, for each  $i = 1, 2, 3, 4$ ,  $A_i$  and  $A_{i+4}$  are buddies due to Observation 1. Let  $m \in \{1, 2\}$  be such that  $B_l$  intersects the line  $L_m$ . Let  $A_{m_1}, A_{m_2}, A_{m_3}$  denote the other three rectangles (from  $\mathcal{A} \cup \mathcal{A}'$ ) which are also intersected by  $L_m$ . Thus we have that  $w(A_{m_1}) + w(A_{m_2}) + w(A_{m_3}) + w(B_l) = 1$ . None of  $A_{m_1}, A_{m_2}, A_{m_3}$  can lie in  $\mathcal{A}'$  because otherwise  $w(A_{m_1}) + w(A_{m_2}) + w(A_{m_3}) + w(B_l) > (\frac{1}{4} + 8\delta) + (\frac{1}{4} + \delta) + 2(\frac{1}{4} - 4\delta) = 1 + \delta$ , a contradiction. Hence  $\{A_{m_1}, A_{m_2}, A_{m_3}\} \subseteq \mathcal{A}$ , and using Lemma 1 we get that  $\{A_{m_1}, A_{m_2}, A_{m_3}\} = \{A_{X,i}, A_{Y,j}, A_{Z,k}\}$  for integers  $i, j, k$  such that  $t_l = (x_i, y_j, z_k)$ , where  $t_l$  is the corresponding triple for the rectangle  $B_l$ . This completes the proof.  $\square$

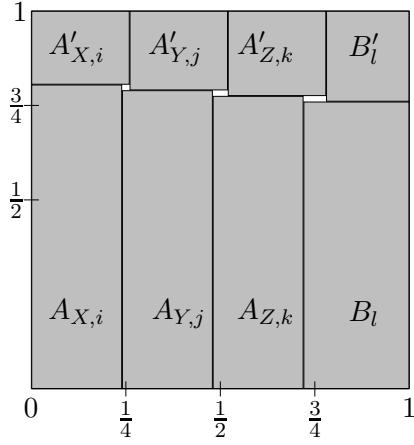


Figure 1.

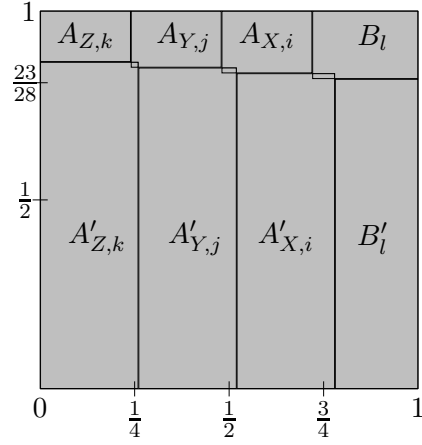


Figure 2.

Now we can prove the main theorem of this section

**Theorem 1** *There is a constant  $\rho > 1$  such that it is NP-hard to approximate 2-DIMENSIONAL BIN PACKING WITH ROTATIONS into unit square bins with an asymptotic approximation ratio less than  $\rho$ .*

*Proof.* Recall that the Bin Packing reduction started from a set  $\mathcal{T}$  of instances of MAX-3DM such that for  $T \in \mathcal{T}$  it is NP-hard to decide of whether  $\text{OPT}(T) \geq \beta(T)$ , or  $\text{OPT}(T) < \alpha(T)$  for some fixed efficiently computable functions  $\alpha, \beta$ .

(a) Assume first that  $T \in \mathcal{T}$  is such that  $\text{OPT}(T) \geq \beta(T)$ . We will show that the collection of rectangles  $\mathcal{R}_T$  from the Bin Packing reduction has its optimum  $\text{OPT}'(\mathcal{R}_T)$  of size at most  $|T| + n - 3\beta(T)$ . Consider a matching  $M$  in  $T$  consisting of  $\beta(T)$  triples. For each triple  $t_l = (x_i, y_j, z_k) \in M$  we create a well-packed bin with rectangles  $\{A_{X,i}, A_{Y,j}, A_{Z,k}, B_l, A'_{X,i}, A'_{Y,j}, A'_{Z,k}, B'_l\}$  packed. For each  $t_l \in T \setminus M$  we can put  $B_l$  and  $B'_l$  along with a dummy rectangle into a bin; in this way we use  $|T| - \beta(T)$  dummy rectangles.

For each of  $n - 3\beta(T)$  elements in  $X \cup Y \cup Z$  that are not covered by  $M$ , we put in a bin the corresponding buddies  $A$  and  $A'$  along with one dummy rectangle. The rest of dummy rectangles is used in this way and all rectangles from  $\mathcal{R}_T$  are packed into  $|T| + n - 3\beta(T)$  bins.

(b) Assume now that  $T \in \mathcal{T}$  satisfies  $\text{OPT}(T) < \alpha(T)$ . Our aim is to estimate  $\text{OPT}'(\mathcal{R}_T)$  from below. Let any feasible solution of 2-BP<sup>r</sup> for an instance  $\mathcal{R}_T$  be fixed from now on. There will be exactly  $N_d = |T| + n - 4\beta(T)$  bins with dummy rectangles, each of them can contain at most one rectangle from  $\mathcal{A} \cup \mathcal{B}$ . Let us consider now bins without dummy rectangles. If such bin is not well-packed then it either contains at most three rectangles from  $\mathcal{A} \cup \mathcal{B}$  or else it contains at most three rectangles from  $\mathcal{A}' \cup \mathcal{B}'$ . Let  $N_g$  denote the number of well-packed bins. Among the bins without dummy rectangles which are not well-packed, let  $N_{b_2}$  denote the number of bins with at most three rectangles from  $\mathcal{A} \cup \mathcal{B}$ , and let  $N_{b_1}$  denote the number of the rest rectangles (i.e.,  $N_{b_1}$  is the number of bins with four rectangles from  $\mathcal{A} \cup \mathcal{B}$ , but with at most three rectangles from  $\mathcal{A}' \cup \mathcal{B}'$ ).

Since all  $|T| + n$  rectangles from  $\mathcal{A} \cup \mathcal{B}$  have to be packed, we have the constraint that

$$4N_g + 4N_{b_1} + 3N_{b_2} + N_d \geq |T| + n,$$

or equivalently

$$4N_g + 4N_{b_1} + 3N_{b_2} \geq 4\beta(T). \quad (4)$$

Recall that rectangles from  $\mathcal{A} \cup \mathcal{B}$  are roughly  $(\frac{1}{4}, \frac{3}{4})$  each, and those from  $\mathcal{A}' \cup \mathcal{B}'$  are roughly  $(\frac{1}{4}, \frac{1}{4})$  each. In what follows we will count rectangles from  $\mathcal{A} \cup \mathcal{B}$  with weight 3, and those from  $\mathcal{A}' \cup \mathcal{B}'$  with weight 1 each. Easy area estimate shows that the total weight of rectangles packed to a unit bin cannot exceed 16. Further, any bin containing a dummy rectangle can contain rectangles from  $\mathcal{A} \cup \mathcal{B} \cup \mathcal{A}' \cup \mathcal{B}'$  of weight at most 4. Observe that each of  $N_{b_1}$  bins contains rectangles of weight at most 15. Hence the second constraint derived from the fact that all rectangles have to be packed reads as follows:

$$16N_g + 15N_{b_1} + 16N_{b_2} + 4N_d \geq 4(|T| + n).$$

Using  $N_d = |T| + n - 4\beta(T)$  and adding the constraint (4) to the last one we get

$$20N_g + 19N_{b_1} + 19N_{b_2} \geq 20\beta(T).$$

Since the set of well-packed bins corresponds to a feasible solution for a MAX-3DM instance  $T$  (by Lemma 5),  $N_g < \alpha(T)$ . Thus, assuming  $\text{OPT}(T) < \alpha(T)$  we get

$$\text{OPT}'(\mathcal{R}_T) > N_g + N_{b_1} + N_{b_2} + N_d \geq \frac{20}{19}\beta(T) - \frac{1}{19}N_g + N_d > |T| + n - 3\beta(T) + \frac{1}{19}(\beta(T) - \alpha(T)).$$

It easily follows that the Bin Packing reduction is a gap preserving reduction assuming that we started from  $(\alpha(T), \beta(T))$ -gap version of the bounded MAX-3DM problem.

Now suppose that for a fixed constant  $\rho$ ,  $1 < \rho < 1 + \frac{1}{19} \frac{\beta(T) - \alpha(T)}{|T| + n - 3\beta(T)}$ , there exists a polynomial time algorithm  $\mathcal{A}_\rho$  and a constant  $C$  such that for instances  $\mathcal{R}_T$  if  $\text{OPT}'(\mathcal{R}_T) > C$ , then  $\mathcal{A}_\rho \leq \rho \text{OPT}'(\mathcal{R}_T)$ . Thus, for any corresponding instance  $T$  of MAX-3DM we could distinguish whether  $\text{OPT}(T) \geq \beta(T)$ , or  $\text{OPT}(T) < \alpha(T)$ , which is an NP-hard problem. Hence, it is NP-hard to achieve an asymptotic approximation ratio  $\leq \rho$  for the problem 2-DIMENSIONAL BIN PACKING WITH ROTATIONS into unit square bins.  $\square$

Using the NP-hard gap result from Theorem A we can obtain an explicit lower bound  $1 + \frac{1}{3792}$  on asymptotic approximation ratio of any polynomial time approximation algorithm for 2-DIMENSIONAL BIN PACKING WITH ROTATIONS into unit bins. For the same problem *without* rotations our method provides a lower bound  $1 + \frac{1}{2196}$ .

## 4 Two-dimensional Bin Covering Problems

In this section we prove non-existence of an APTAS for both versions of the 2-DIMENSIONAL BIN COVERING problems without and with ninety-degree rotations allowed, respectively. Our gap preserving reduction from the bounded MAX-3DM problem was presented in Section 2 along with its basic properties. Even if our analysis has some similarities with that given for packings, the case of coverings appears to be technically more complicated to handle. One of reasons that makes the case of packings easier, is that for packing we have à priori an upper bound on number of rectangles used for a single bin. For covering problems no such upper bound is available and we have to deal with a variety of possibilities how a bin can be covered.

First we start with a simple observation about the sizes of rectangles of  $\mathcal{R}_T$  used in the Bin Covering reduction.

**Observation 3** *For any rectangle  $A \in \mathcal{A}$  implies  $w(A) + h(A) = \frac{3}{7} - 5\delta$ ,  $A' \in \mathcal{A}'$  implies  $w(A') + h(A') = \frac{15}{14} + 5\delta$ ,  $B \in \mathcal{B}$  implies  $w(B) + h(B) = \frac{3}{7} + 8\delta$ , and  $B' \in \mathcal{B}'$  implies  $w(B') + h(B') = \frac{15}{14} - 8\delta$ .*

In the following we will derive some properties of certain 8-tuples of rectangles from  $\mathcal{A} \cup \mathcal{B} \cup \mathcal{A}' \cup \mathcal{B}'$  covering completely a unit square bin. This analysis will be used later in the proofs of Lemma 7 and Theorem 2.

**Analysis.** Consider an oriented covering of the bin  $[0, 1]^2$  by exactly four rectangles from  $\mathcal{A} \cup \mathcal{B}$  and four rectangles from  $\mathcal{A}' \cup \mathcal{B}'$ . Then no two of points  $[0, 0]$ ,  $[\frac{1}{3}, 0]$ ,  $[\frac{2}{3}, 0]$ ,  $[1, 0]$ ,  $[0, 1]$ ,  $[\frac{1}{3}, 1]$ ,  $[\frac{2}{3}, 1]$ , and  $[1, 1]$  belong to the same rectangle. Let  $A_1, A_2, A_3, \dots, A_8$  denote the rectangles that contain these points, respectively. Clearly,  $A_1 \cup A_2 \cup A_3 \cup A_4$  covers  $\{(x, 0) : x \in [0, 1]\}$ , hence

$$\sum_{i=1}^4 w(A_i) \geq 1. \quad (5)$$

Similarly,  $A_5 \cup A_6 \cup A_7 \cup A_8$  covers  $\{(x, 1) : x \in [0, 1]\}$ , hence

$$\sum_{i=1}^4 w(A_{i+4}) \geq 1. \quad (6)$$

For each  $i = 1, 2, 3, 4$ , the set  $(A_i \cup A_{i+4})$  is a small (depending on  $\delta$ ) perturbation of  $[\frac{i-1}{4}, \frac{i}{4}] \times [0, 1]$ . In particular, the segment  $\{(\frac{i-1}{3}, y) : y \in [0, 1]\}$  is covered by  $A_i \cup A_{i+4}$  only and hence

$$h(A_i) + h(A_{i+4}) \geq 1 \quad \text{for } i = 1, 2, 3, 4. \quad (7)$$

Consequently, for each  $i \in \{1, 2, 3, 4\}$  at least one of rectangles  $A_i, A_{i+4}$  belongs to  $\mathcal{A}' \cup \mathcal{B}'$ . But as exactly four rectangles are from  $\mathcal{A}' \cup \mathcal{B}'$ , it easily follows that for each  $i \in \{1, 2, 3, 4\}$  exactly one of rectangles  $A_i, A_{i+4}$  belongs to  $\mathcal{A} \cup \mathcal{B}$ , and one to  $\mathcal{A}' \cup \mathcal{B}'$ . As height of any rectangle from  $\mathcal{A} \cup \mathcal{B}$  is less than  $\frac{1}{2}$ , the segment  $\{(x, \frac{1}{2}) : x \in [0, 1]\}$  is covered by rectangles from  $\mathcal{A}' \cup \mathcal{B}'$ . Thus

$$\sum_{\substack{i=1 \\ A_i \in \mathcal{A}' \cup \mathcal{B}'}}^8 w(A_i) \geq 1. \quad (8)$$

Inspecting the range of heights of rectangles in  $\mathcal{A}_X, \mathcal{A}_Y, \mathcal{A}_Z, \mathcal{B}, \mathcal{A}'_X, \mathcal{A}'_Y, \mathcal{A}'_Z, \mathcal{B}'$  leads to more restrictions on possible combinations in pairs  $A_i, A_{i+4}$ . We will employ an observation that if one of  $A_i, A_{i+4}$  belongs to  $\mathcal{B}'$  then the another one belongs to  $\mathcal{B}$ . In particular,

$$|\mathcal{B} \cap \{A_i : 1 \leq i \leq 8\}| \geq |\mathcal{B}' \cap \{A_i : 1 \leq i \leq 8\}|. \quad (9)$$

We can observe further that

$$|\mathcal{B}' \cap \{A_i : 1 \leq i \leq 8\}| \leq 1. \quad (10)$$

To show (10), let  $j := |\mathcal{B}' \cap \{A_i : 1 \leq i \leq 8\}|$ . Thus  $j$  rectangles from  $A_1, \dots, A_8$  belong to  $\mathcal{B}'$  and  $(4 - j)$  belong to  $\mathcal{A}'$ , thus  $\sum_{i=1, A_i \in \mathcal{A}' \cup \mathcal{B}'}^8 w(A_i) < 1 + 16\delta - 12j\delta$ . Due to (8), only  $j = 0$  or  $j = 1$  are possible values for an integer  $j$ .

Now we notice that even if rotations are allowed, for some important sets of rectangles the only possibility how to cover a bin is, in fact, without using rotations.

**Lemma 6** *Suppose that an  $r$ -covering of a unit square bin consists of a set  $\mathcal{C}$  of eight rectangles from which four are from  $\mathcal{A} \cup \mathcal{B}$  and four are from  $\mathcal{A}' \cup \mathcal{B}'$ . Then either all eight rectangles are placed in the bin in the initial orientation, or all rotated by ninety degrees. Moreover,  $|\mathcal{B} \cup \mathcal{C}| \geq |\mathcal{B}' \cup \mathcal{C}|$ .*

*Proof.* It is easy to estimate that the total area of four rectangles from  $\mathcal{A} \cup \mathcal{B}$  and four rectangles from  $\mathcal{A}' \cup \mathcal{B}'$  is less than  $1 + 30\delta$ . Hence, in an  $r$ -covering of a unit square bin by rectangles from  $\mathcal{C}$  the area with multiple covering is less than  $30\delta$ . Thus it is sufficient to show that if rectangles fail to be oriented in the same way in the given  $r$ -covering (i.e., either all in the initial orientation, or all rotated by ninety degrees), then their overlap is larger.

We show first that those rectangles of  $\mathcal{C}$  that belong to  $\mathcal{A}' \cup \mathcal{B}'$  are oriented in such  $r$ -covering in the same way. If two such rectangles are oriented differently, then it is easy to see that they overlap in a rectangle with both sides larger than  $(\frac{1}{14} - 8\delta)$ . Thus, the overlap area is larger than  $\frac{1}{196} - \frac{8}{7}\delta$ , which is larger than  $30\delta$ . This contradiction shows that the rectangles of  $\mathcal{C}$  belonging to  $\mathcal{A}' \cup \mathcal{B}'$  are oriented in the same way. We can assume from now on that these four rectangles are placed in the bin in the initial orientation, the case when all are rotated can be discussed in the same way.

To prove that then *all* rectangles from  $\mathcal{C}$  are placed in the initial orientation assume, on the contrary, that a rectangle  $A \in (\mathcal{A} \cup \mathcal{B}) \cap \mathcal{C}$  is rotated by ninety degrees. It is easy to see that any segment  $\{y\} \times [0, 1]$  ( $y \in [0, 1]$ ) has to intersect a rectangle from  $(\mathcal{A}' \cup \mathcal{B}') \cap \mathcal{C}$ . Consider any  $y \in [0, 1]$  such that  $\{y\} \times [0, 1]$  intersects  $A$ . (The segment of such  $y$ 's has length greater than  $\frac{5}{28} - 4\delta$ .) Now  $\{y\} \times [0, 1]$  intersects  $A$  in a segment  $I(y)$  larger than  $\frac{1}{4} - \delta$  and it intersects some rectangle from  $(\mathcal{A}' \cup \mathcal{B}') \cap \mathcal{C}$  in a segment  $I'(y')$  larger than  $\frac{23}{28}$ . Thus  $I(y) \cap I'(y)$  is larger than  $\frac{1}{14} - 4\delta$ . Consequently,  $A$  intersects the union of  $(\mathcal{A}' \cup \mathcal{B}') \cap \mathcal{C}$  in a set of area greater than  $(\frac{1}{14} - 4\delta)(\frac{5}{28} - 4\delta) > \frac{5}{392} - \delta$ . Thus the overlap is larger than  $30\delta$ ; a contradiction showing that all rectangles of  $\mathcal{C}$  are oriented in the same way. Now we apply (9) which implies that  $|\mathcal{B} \cap \mathcal{C}| \geq |\mathcal{B}' \cap \mathcal{C}|$ .  $\square$

In the next step we introduce the notion of *well-covered bins* by rectangles of  $\mathcal{R}_T$  and characterize them in terms of an instance  $T$  of MAX-3DM.

**Definition 6** *Given an  $r$ -covering of a bin by some rectangles from  $\mathcal{A} \cup \mathcal{B} \cup \mathcal{A}' \cup \mathcal{B}'$ . The bin is called well-covered if it is covered by exactly eight rectangles from which four are from  $\mathcal{A} \cup \mathcal{B}$ , four are from  $\mathcal{A}' \cup \mathcal{B}'$ , and the number of rectangles from  $\mathcal{B}$  is the same as those from  $\mathcal{B}'$ .*

**Lemma 7** *A bin can be well-covered by a given 8-tuple  $\mathcal{C}$  of rectangles if and only if  $\mathcal{C}$  consists of rectangles  $A_{X,i}$ ,  $A_{Y,j}$ ,  $A_{Z,k}$ ,  $B_l$ ,  $A'_{X,i}$ ,  $A'_{Y,j}$ ,  $A'_{Z,k}$ , and  $B'_l$  for some  $t_l = (x_i, y_j, z_k) \in T$ .*

*Proof.* We show first that an 8-tuple of rectangles that correspond to a triple  $t_l = (x_i, y_j, z_k) \in T$  in a way as described above can cover a unit square bin even without using rotations.

Starting from the bottom left corner of the bin and moving towards the right, each of rectangles  $A'_{Z,k}$ ,  $A'_{Y,j}$ ,  $A'_{X,i}$ , and  $B'_l$ , (in this order) is placed such that it touches the bottom of the bin  $\mathbb{B}$  and the previous rectangle (see Fig. 2). The remaining rectangles  $A_{Z,k}$ ,  $A_{Y,j}$ ,  $A_{X,i}$ , and  $B_l$  will be placed in this order starting from the top left corner of the bin and moving towards the right, such that each rectangle touches the top of the bin and the previous rectangle. Clearly, these four rectangles cover the gap left in the bin after the first four rectangles were placed, as

$$\begin{aligned} w(A_{Z,k}) + w(A_{Y,j}) + w(A_{X,i}) + w(B_l) &= 1, \\ h(A_{Z,k}) + h(A'_{Z,k}) &= h(A_{Y,j}) + h(A'_{Y,j}) = h(A_{X,i}) + h(A'_{X,i}) = h(B_l) + h(B'_l) = 1, \\ h(A'_{Z,k}) &> h(A'_{Y,j}) > h(A'_{X,i}) > h(B'_l), \\ w(A_{Z,k}) < w(A'_{Z,k}), w(A_{Y,j}) < w(A'_{Y,j}), \text{ and } w(A_{X,i}) < w(A'_{X,i}). \end{aligned}$$

Assume now that a bin is well-covered by rectangles of  $\mathcal{C}$ . Lemma 6 implies, in particular, that either all rectangles of  $\mathcal{C}$  are placed in the initial orientation, or all are rotated by ninety degrees. We will assume that the former case occurs, the latter one can be discussed in the same way.

Let  $A_1, A_2, \dots, A_8$  denote rectangles of  $\mathcal{C}$  that cover the points  $[0, 0]$ ,  $[\frac{1}{3}, 0]$ ,  $[\frac{2}{3}, 0]$ ,  $[1, 0]$ ,  $[0, 1]$ ,  $[\frac{1}{3}, 1]$ ,  $[\frac{2}{3}, 1]$ , and  $[1, 1]$ , respectively. Recall that from the analysis above (5), (6), and (7) hold.

As  $|\mathcal{B} \cap \{A_i : 1 \leq i \leq 8\}| = |\mathcal{B}' \cap \{A_i : 1 \leq i \leq 8\}|$  (the bin is well-covered), we have  $\sum_{i=1}^8 (w(A_i) + h(A_i)) = 6$  by Observation 3, and in each of inequalities (5), (6) and (7) equality must hold. Therefore  $A_i$  and  $A_{i+4}$  are buddies for each  $i \in \{1, 2, 3, 4\}$  by Observation 1. Hence, in particular, no two rectangles among  $A_1, A_2, A_3, A_4$  are buddies. Now from  $\sum_{i=1}^4 w(A_i) = 1$  and Lemma 2 it follows that  $|(\mathcal{B} \cup \mathcal{B}') \cap \{A_i : 1 \leq i \leq 8\}| > 0$ . Thus  $|\mathcal{B} \cap \{A_i : 1 \leq i \leq 8\}| = |\mathcal{B}' \cap \{A_i : 1 \leq i \leq 8\}|$  and (10) implies that  $|\mathcal{B} \cap \{A_i : 1 \leq i \leq 8\}| = |\mathcal{B}' \cap \{A_i : 1 \leq i \leq 8\}| = 1$ . Hence  $\{A_1, A_2, \dots, A_8\}$  consists of buddies  $B_l \in \mathcal{B}$ ,  $B'_l \in \mathcal{B}'$  for some  $l \in \{1, 2, \dots, |T|\}$ , and six rectangles (3 pairs of buddies) from  $\mathcal{A} \cup \mathcal{A}'$ . Let us assume that  $B'_l \in \{A_5, A_6, A_7, A_8\}$  (i.e.,  $B_l \in \{A_1, A_2, A_3, A_4\}$ ); the opposite case can be discussed, due to the symmetry, in a similar way.

Recall that  $w(B'_l) < \frac{1}{4} - 8\delta$ ,  $\sum_{i=5}^8 w(A_i) = 1$ , and  $\{A_5, A_6, A_7, A_8\} \setminus \{B'_l\} \subseteq \mathcal{A} \cup \mathcal{A}'$ . If some rectangle of  $A_5, A_6, A_7, A_8$  belongs to  $\mathcal{A}$ , we easily get

$$\sum_{i=5}^8 w(A_i) < (\frac{1}{4} - 8\delta) + (\frac{1}{4} - \delta) + 2(\frac{1}{4} + 4\delta) = 1 - \delta,$$

a contradiction. Thus  $\{A_5, A_6, A_7, A_8\} \setminus \{B'_l\} \subseteq \mathcal{A}'$ , consequently  $\{A_1, A_2, A_3, A_4\} \setminus \{B_l\} \subseteq \mathcal{A}$ . Recalling  $\sum_{i=1}^4 w(A_i) = 1$ , Lemma 1 implies that  $\{A_1, A_2, A_3, A_4\} = \{A_{X,i}, A_{Y,j}, A_{Z,k}, B_l\}$  for some  $i, j, k, l$  such that  $(x_i, y_j, z_k) = t_l$ .

This completes the proof.  $\square$

Now we are ready to prove the main result of this paper.

**Theorem 2** *Unless  $P = NP$ , there is no APTAS for the problems 2-DIMENSIONAL BIN COVERING and 2-DIMENSIONAL BIN COVERING WITH ROTATIONS.*

*Proof.* The Bin Covering reduction starts from a set  $\mathcal{T}$  of instances of MAX-3DM such that for an instance  $T \in \mathcal{T}$  it is NP-hard to decide of whether  $\text{OPT}(T) \geq \beta(T)$ , or  $\text{OPT}(T) < \alpha(T)$  for some fixed efficiently computable functions  $\alpha, \beta$ . For any  $T \in \mathcal{T}$ , the reduction defines a collection of rectangles  $\mathcal{R}_T = \mathcal{A} \cup \mathcal{A}' \cup \mathcal{B} \cup \mathcal{B}' \cup \mathcal{D}$ , let  $\text{OPT}'(\mathcal{R}_T)$  (resp.  $\text{OPT}''(\mathcal{R}_T)$ ) denote the corresponding optima of 2-BC (resp. 2-BC<sup>r</sup>) for an instance  $\mathcal{R}_T$ .

We start with the proof of the following two implications which describe how the NP-hard gap for MAX-3DM is preserved by the Bin Covering reduction to similar NP-hard gaps for 2-BC and 2-BC<sup>r</sup>.

(A) If  $\text{OPT}(T) \geq \beta(T)$  then

$$\text{OPT}'(\mathcal{R}_T) \geq |T| + n - 3\beta(T).$$

(B) If  $\text{OPT}(T) < \alpha(T)$  then

$$\text{OPT}''(\mathcal{R}_T) < (1 - \varepsilon(T))(|T| + n - 3\beta(T)),$$

$$\text{where } \varepsilon(T) = \frac{1}{897} \frac{\beta(T) - \alpha(T) - 6}{|T| + n - 3\beta(T)}.$$

*The proof of (A).*

Let  $T \in \mathcal{T}$  satisfying  $\text{OPT}(T) \geq \beta(T)$  be fixed. Consider a matching  $M$  in  $T$  consisting of  $\beta(T)$  triples. For each triple  $t_l = (x_i, y_j, z_k) \in M$ , the corresponding 8 rectangles  $\{A_{X,i}, A_{Y,j},$

$A_{Z,k}, B_l, A'_{X,i}, A'_{Y,j}, A'_{Z,k}, B'_l\}$  will cover one (well-covered) bin. For each  $t_l \in T \setminus M$  we take  $B_l, B'_l$ , and a dummy rectangle to cover a bin. In this way we use  $|T| - \beta(T)$  dummy rectangles.

For each of  $n - 3\beta(T)$  elements in  $X \cup Y \cup Z$  that are not contained in triples of  $M$  and for remaining  $n - 3\beta(T)$  dummy rectangles we take the corresponding buddies  $A, A' \in \mathcal{A} \cup \mathcal{A}'$  along with a dummy rectangle to cover a bin. Hence, we have covered  $\beta(T) + (|T| - \beta(T)) + (n - 3\beta(T)) = |T| + n - 3\beta(T)$  bins in total and  $\text{OPT}'(\mathcal{R}_T) \geq |T| + n - 3\beta(T)$  follows.

*The proof (B).*

Let  $T \in \mathcal{T}$  satisfying  $\text{OPT}(T) < \alpha(T)$  be fixed and consider any optimal solution of 2-BC<sup>r</sup> for an instance  $\mathcal{R}_T$  with  $\text{OPT}''(\mathcal{R}_T)$  covered bins. To simplify some considerations we first normalize the solution without decreasing the number of covered bins as follows:

- (i) *Each bin is covered using at most one dummy rectangle.* It is easy to see that in an optimal solution we have more than  $|T| + n - 4\beta(T)$  bins covered. Thus, if two dummy rectangles are used to cover the same bin, we can take another bin that is covered without dummy rectangle and change the covering of these two bins such that each of them uses one of dummy rectangles. (In the covering of a bin without dummy rectangles one of the following possibilities has to appear: (a) at least four rectangles are from  $\mathcal{A}' \cup \mathcal{B}'$ , (b) three rectangles are from  $\mathcal{A}' \cup \mathcal{B}'$  and at least three rectangles are from  $\mathcal{A} \cup \mathcal{B}$ , (c) at least 10 rectangles are from  $\mathcal{A} \cup \mathcal{B}$ . Clearly, in all three cases the rectangles can be partitioned into two sets such that each set along with a dummy rectangle can cover a unit square bin.)
- (ii) If rectangles  $A_1, A_2, \dots, A_j$  cover a bin, then no proper subset of them can cover it.
- (iii) To ensure (ii), some of rectangles (the *rest*) can be left unused, but it is impossible to cover a bin by the *rest*. One can ensure that no dummy rectangle is in the *rest* (the discussion is similar to (i)). The *rest* can contain at most six rectangles from  $\mathcal{A}' \cup \mathcal{B}'$ .

Now let a normalized optimal solution for  $\mathcal{R}_T$  be fixed. Recall that rectangles from  $\mathcal{A} \cup \mathcal{B}, \mathcal{A}' \cup \mathcal{B}'$ , and  $\mathcal{D}$  are small perturbations of rectangles  $(\frac{1}{4}, \frac{5}{28}), (\frac{1}{4}, \frac{23}{28}),$  and  $(\frac{3}{4}, 1)$ , respectively. In our counting arguments we will assign them weights  $\frac{5}{112}, \frac{23}{112},$  and  $\frac{3}{4}$ , which almost precisely correspond to their respective areas. It is easy to observe that any covered bin uses rectangles of total weight at least 1. Therefore the total weight of all rectangles,  $|T| + n - 3\beta(T)$ , is a trivial upper bound on the number of covered bins. To achieve this bound one has to cover each bin by rectangles of weight exactly 1. It turns out that in case  $\text{OPT}(T) < \alpha(T)$  this is not possible, and we will necessarily have a significant portion of bins covered by rectangles of weight strictly larger than 1.

Among covered bins we will distinguish bins of several kinds:

- (a) *D-bins* – the bins that use a dummy rectangle in their covering and their number is  $N_d = |T| + n - 4\beta(T)$ ,
- (b) the remaining covered bins are termed *non-D-bins*.
  - (a) Firstly we will consider *D-bins*. Let  $N_{d_0}$  be the number of *D-bins* which use one rectangle from  $\mathcal{A}' \cup \mathcal{B}'$  and one from  $\mathcal{A} \cup \mathcal{B}$ . Observe that if one of those rectangles belongs to  $\mathcal{B}'$  then the another one belongs to  $\mathcal{B}$ , otherwise the bin is not covered. (This fact will be used later.) Each of these *D-bins* is covered by rectangles of total weights 1. A set of rectangles (except dummy rectangle) covering each of the remaining  $N_d - N_{d_0}$  *D-bins* is one of the following types according to what rectangles they use:

- none of  $\mathcal{A}' \cup \mathcal{B}'$  and six rectangles from  $\mathcal{A} \cup \mathcal{B}$ ; total weight is  $\frac{57}{56}$ ,
- one rectangle from  $\mathcal{A}' \cup \mathcal{B}'$  and two rectangles from  $\mathcal{A} \cup \mathcal{B}$ ; total weight is  $\frac{117}{112}$ ,
- two rectangles from  $\mathcal{A}' \cup \mathcal{B}'$  and none from  $\mathcal{A} \cup \mathcal{B}$ ; total weight is  $\frac{65}{56}$ .

We can summarize that each of the above  $N_d - N_{d_0}$   $D$ -bins uses rectangles of total weight at least  $\frac{57}{56}$ .

(b) Now we describe how non- $D$ -bins are distinguished. Let  $N_g$  be the number of well-covered bins. Due to Lemma 7,  $N_g \leq \text{OPT}(T) < \alpha(T)$ . Let further  $N_b$  be the number of bins covered by four rectangles from  $\mathcal{A}' \cup \mathcal{B}'$  and four rectangles from  $\mathcal{A} \cup \mathcal{B}$ , but which are not well-covered. Due to Lemma 6 and 7,

$$\text{each of these } N_b \text{ bins uses strictly more rectangles from } \mathcal{B} \text{ than from } \mathcal{B}'. \quad (\star)$$

Each of the above  $N_g + N_b$  non- $D$ -bins is covered by rectangles of total weight 1. Let  $N_{b_1}$  be the number of the remaining non- $D$ -bins. In the following we observe that each of them is covered by rectangles of total weight at least  $\frac{113}{112}$ . To see that, consider such a bin covered by  $k$  rectangles from  $\mathcal{A}' \cup \mathcal{B}'$  and  $l$  rectangles from  $\mathcal{A} \cup \mathcal{B}$  such that  $(k, l) \neq (4, 4)$ . As the total weight  $\frac{23}{112}k + \frac{5}{112}l$  has to be at least 1, it in turn implies that this weight is at least  $\frac{113}{112}$ .

From the above considerations (a)-(b) it follows that rectangles covering bins have total weight at least

$$N_g + N_b + \frac{113}{112}N_{b_1} + N_{d_0} + \frac{57}{56}(N_d - N_{d_0}).$$

On the other hand, the total weight of all rectangles is  $|T| + n - 3\beta(T)$ , hence

$$N_g + N_b + \frac{113}{112}N_{b_1} + N_d + \frac{1}{56}(N_d - N_{d_0}) \leq |T| + n - 3\beta(T).$$

As  $N_d = |T| + n - 4\beta(T)$ , after multiplying by 112 the above reads as

$$112N_g + 112N_b + 113N_{b_1} \leq 112\beta(T) - 2(N_d - N_{d_0}).$$

We can rewrite the last inequality in two different ways:

$$112(N_g + N_b + N_{b_1}) \leq 112\beta(T) - 2(N_d - N_{d_0}) - N_{b_1}, \quad \text{and} \quad (11)$$

$$113(N_g + N_b + N_{b_1}) \leq 113\beta(T) - (\beta(T) - N_g) + N_b - 2(N_d - N_{d_0}). \quad (12)$$

We can simplify (11) to

$$N_g + N_b + N_{b_1} \leq \beta(T) - \frac{1}{112}N_{b_1},$$

and use  $N_g < \alpha(T)$  in (12) to obtain

$$N_g + N_b + N_{b_1} < \beta(T) - \frac{1}{113}(\beta(T) - \alpha(T)) + \frac{1}{113}(N_b - 2(N_d - N_{d_0})).$$

Thus  $\text{OPT}''(\mathcal{R}_T)$  ( $= N_g + N_b + N_{b_1} + N_d$ ) is estimated in two different ways

$$\text{OPT}''(\mathcal{R}_T) \leq |T| + n - 3\beta(T) - \frac{1}{112}N_{b_1}, \quad (13)$$

$$\text{OPT}''(\mathcal{R}_T) < |T| + n - 3\beta(T) - \frac{1}{113}(\beta(T) - \alpha(T)) + \frac{1}{113}(N_b - 2(N_d - N_{d_0})). \quad (14)$$

To show that at least one of inequalities (13) and (14) completes the proof of (B), we now derive one more constraint.

For our fixed normalized solution let  $S$  denote in what follows either a set of rectangles that cover a single bin, or the *rest*. Let  $\mathcal{S}$  be the collection of all such sets  $S$ . For each set  $S \in \mathcal{S}$  we define  $\varphi(S)$  to be the number  $\varphi(S) = |S \cap \mathcal{B}| - |S \cap \mathcal{B}'|$ . As  $|\mathcal{B}| = |\mathcal{B}'|$  and elements of  $\mathcal{S}$  define



a partition of all rectangles,  $\sum_{S \in \mathcal{S}} \varphi(S) = 0$  follows. If  $S$  corresponds to a well-covered bin then  $\varphi(S) = 0$ , and at least for  $N_b$  non- $D$ -bins we have  $\varphi(S) \geq 1$  (due to the property  $(\star)$ ). For the remaining sets  $S \in \mathcal{S}$  we estimate  $\varphi(S)$  from below. As any set of 7 rectangles from  $\mathcal{A}' \cup \mathcal{B}'$  can cover a bin,  $\varphi(S) \geq -7$  always holds. Moreover,  $\varphi(S) \geq -6$  if  $S$  is the *rest* (the property (iii) of the normalized solution). At least for  $N_{d_0}$   $D$ -bins we have  $\varphi(S) \geq 0$  for the corresponding set  $S$ . For a set  $S$  corresponding to any of remaining  $(N_d - N_{d_0})$   $D$ -bins we have  $\varphi(S) \geq -2$ , as a  $D$ -bin uses at most two rectangles from  $\mathcal{A}' \cup \mathcal{B}'$  in its covering. Hence we have proved

$$0 = \sum_{S \in \mathcal{S}} \varphi(S) \geq N_b - 7N_{b_1} - 2(N_d - N_{d_0}) - 6$$

that reads as

$$N_b - 2(N_d - N_{d_0}) \leq 7N_{b_1} + 6.$$

Using the last inequality we can rewrite (14) in the form

$$\text{OPT}''(\mathcal{R}_T) < |T| + n - 3\beta(T) - \frac{1}{113}(\beta(T) - \alpha(T)) + \frac{1}{113}(7N_{b_1} + 6). \quad (14')$$

Put  $\varepsilon(T) := \frac{1}{897} \frac{\beta(T) - \alpha(T) - 6}{|T| + n - 3\beta(T)}$ . We will distinguish the following two cases:

(I) Assume first that  $\frac{1}{112}N_{b_1} > \frac{1}{897}(\beta(T) - \alpha(T) - 6)$ . Then (13) implies that

$$\text{OPT}''(\mathcal{R}_T) < |T| + n - 3\beta(T) - \frac{1}{897}(\beta(T) - \alpha(T) - 6) = (1 - \varepsilon(T))(|T| + n - 3\beta(T)).$$

(II) Assume now that  $\frac{1}{112}N_{b_1} \leq \frac{1}{897}(\beta(T) - \alpha(T) - 6)$ . Then (14') implies that

$$\begin{aligned} \text{OPT}''(\mathcal{R}_T) &< |T| + n - 3\beta(T) - \frac{1}{113}(\beta(T) - \alpha(T) - 6) + \frac{7}{113} \frac{112}{897}(\beta(T) - \alpha(T) - 6) \\ &= (1 - \varepsilon(T))(|T| + n - 3\beta(T)). \end{aligned}$$

This completes the proof of (B).

As we use parameters  $n = 3q$ ,  $|T| = 2q$ , and  $\alpha(T)$ ,  $\beta(T)$  as in Theorem A, it easily follows that  $\varepsilon(T)$  is bounded from below by a positive constant  $\varepsilon_0$  for all sufficiently large instances  $T \in \mathcal{T}$ . Clearly,  $\text{OPT}''(\mathcal{R}_T) \geq \text{OPT}'(\mathcal{R}_T)$ , as every feasible solution of 2-BC is a feasible solution of 2-BC' as well. Hence it is NP-hard to distinguish of whether  $\text{OPT}''(\mathcal{R}_T) \geq \text{OPT}'(\mathcal{R}_T) \geq |T| + n - 3\beta(T)$ , or  $\text{OPT}'(\mathcal{R}_T) \leq \text{OPT}''(\mathcal{R}_T) < (1 - \varepsilon_0)(|T| + n - 3\beta(T))$ . Consequently, it is NP-hard to achieve an asymptotic approximation ratio smaller than  $\frac{1}{1 - \varepsilon_0}$  for 2-DIMENSIONAL BIN COVERING, and for 2-DIMENSIONAL BIN COVERING WITH ROTATION as well.  $\square$

## 5 Three-dimensional Strip Packing and Covering problems

In this section we apply the approximation hardness results for 2-dimensional bin packing and covering problems to obtain similar hardness results for some variants of 3-dimensional strip packing and covering problems.

Let a list of 2-dimensional rectangles  $\mathcal{L} = \{(w(R^1), h(R^1)), \dots, (w(R^n), h(R^n))\}$  with a bin  $\mathbb{B} = (b_1, b_2)$  be an instance of the 2-DIMENSIONAL BIN PACKING problem (possibly with rotations). For a fixed parameter  $t > 0$  we define an instance of the 3-DIMENSIONAL STRIP PACKING problem (possibly with rotations) as a list of 3-dimensional rectangles  $\mathcal{L}_t = \{(w(R^1), h(R^1), t), \dots, (w(R^n), h(R^n), t)\}$

with a strip  $(b_1, b_2, \infty)$ . The optimum of all three variants of 3-SP, 3-SP<sup>r</sup>, and 3-SP<sup>z</sup>, for the instance  $\mathcal{L}_t$  can be expressed using the optimum for the instance  $\mathcal{L}$  of 2-DIMENSIONAL BIN PACKING (possibly with rotations) as follows.

**Lemma 8** *If  $\text{OPT}(\mathcal{L})$  denote the optimum for an instance  $\mathcal{L}$  of 2-BP and  $\text{OPT}'(\mathcal{L}_t)$  the optimum for the corresponding 3-dimensional instance  $\mathcal{L}_t$  of 3-SP, then  $\text{OPT}'(\mathcal{L}_t) = t \cdot \text{OPT}(\mathcal{L})$ . The same relation holds also between the optimum for an instance  $\mathcal{L}$  of 2-BP<sup>r</sup> and the optimum for the corresponding 3-dimensional instance  $\mathcal{L}_t$  of 3-SP<sup>z</sup>, respectively 3-SP<sup>r</sup> if we additionally assume that  $t > \max\{b_1, b_2\}$ .*

*Proof.* (i) Consider a packing of  $\mathcal{L}$  into  $\text{OPT}(\mathcal{L})$  bins with side-lengths  $(b_1, b_2)$ . It generates a strip packing of  $\mathcal{L}_t$  into a strip  $(b_1, b_2, \infty)$  with height  $t \cdot \text{OPT}(\mathcal{L})$ . Hence  $\text{OPT}'(\mathcal{L}_t) \leq t \cdot \text{OPT}(\mathcal{L})$ . Now assume, that  $\mathcal{L}_t$  can be packed into the strip  $(b_1, b_2, \infty)$  with height  $h < t \cdot \text{OPT}(\mathcal{L})$ . If  $s = \min\{t, t \cdot \text{OPT}(\mathcal{L}) - h\}$ , then planes  $\{z = t - s\}, \{z = 2t - s\}, \dots, \{z = t \cdot (\text{OPT}(\mathcal{L}) - 1) - s\}$  intersect interiors (or touch bottom) of all rectangles from the list  $\mathcal{L}$ . These plane cuts determine packing of  $\mathcal{L}$  into  $\text{OPT}(\mathcal{L}) - 1$  bins with side-lengths  $(b_1, b_2)$ , a contradiction that completes the proof.

The statement for 3-SP<sup>z</sup> can be proved in the same way. Moreover, if  $t > \max\{b_1, b_2\}$ , then any  $r$ -packing of  $\mathcal{L}_t$ -rectangles into the strip  $(b_1, b_2, \infty)$  has to be  $z$ -oriented.  $\square$

Using Lemma 8 it is easy to see that non-existence of an APTAS for 2-BP ([4]) implies non-existence of APTAS for the 3-SP problem, unless  $P = NP$ . Moreover, using a heterogeneous scaling one can obtain from hardness results for 2-BP some inapproximability results also for 3-SP<sup>z</sup> and 3-SP<sup>r</sup> with a strip  $(b, 1, \infty)$ , for any fixed  $b \in (0, \frac{1}{2})$ . For the strip with unit square base we can use an approximation hardness result obtained above for 2-BP<sup>r</sup> with unit square bin.

**Theorem 3** *There is no APTAS for any of 3-dimensional strip packing problems 3-SP, 3-SP<sup>z</sup>, and 3-SP<sup>r</sup> on instances with the strip  $(1, 1, \infty)$ , unless  $P = NP$ .*

*Proof.* The result for 3-SP follows directly from [4] by Lemma 8. Similarly, Theorem 1 together with Lemma 8 imply that no APTAS can exist, unless  $P = NP$ , for 3-SP<sup>z</sup> and 3-SP<sup>r</sup> on instances with the strip  $(1, 1, \infty)$ .  $\square$

In the same way as for packings, the 2-DIMENSIONAL COVERING problem can be seen as particular case of the 3-DIMENSIONAL STRIP COVERING problem. The transformation described above for strip packing problems has similar properties for strip covering problems as well. For an integer  $t > 0$  a transformation transforming an instance  $\mathcal{L}$  of 2-BC (resp., 2-BC<sup>r</sup>) to an instance  $\mathcal{L}_t$  of 3-SC (resp., 3-SC<sup>z</sup>) essentially preserves an optimum value, namely the ratio between the optimum values for 3-SC (resp., 3-SC<sup>z</sup>) and 2-BC (resp., 2-BC<sup>r</sup>) is exactly  $t$ . The proof is very similar to the one given above for packings. Thus approximation hardness results for 2-BC (resp., 2-BC<sup>r</sup>) with unit bin derived in the previous section translates to the same approximation hardness results for 3-SC (resp., 3-SC<sup>z</sup>) with a strip  $(1, 1, \infty)$ .

We can summarize these results as follows

**Theorem 4** *There are no APTAS for strip covering problems 3-SC and 3-SC<sup>z</sup> on instances with the strip  $(1, 1, \infty)$ , unless  $P = NP$ .*

## 6 Maximum Rectangle Packing Problem

Another rectangle bin packing problem well studied in the literature (e.g., [16], [2]) is the following:

**Definition 7** *Given a collection of  $d$ -dimensional rectangles along with a  $d$ -dimensional rectangular bin  $\mathbb{B}$ ,  $d \geq 2$ . The goal of the MAXIMUM  $d$ -DIMENSIONAL RECTANGLE PACKING problem is to pack the maximum number of rectangles from the collection into a single bin  $\mathbb{B}$ .*

The problem is motivated by scheduling parallel jobs with a common due date to maximize the profit of jobs completed by the due date, where each job can require several processors which are allocated on a line. It has also applications in the advertisement placement problem, see [12] for more details. Other variants of this problem are studied as well, for example, each of rectangles can be associated with weight, and the goal is to maximize the total weight of packed rectangles. In some variants ninety-degree rotations of rectangles can be allowed. But even in the simplest case, 2-dimensional unweighted case without rotations, only a  $(2 + \varepsilon)$ -approximation algorithm is known [16]. The question of whether there is an APTAS is open. However, already in 3-dimensional case the problem can be settled in the negative.

**Theorem 5** *Unless  $P = NP$ , there is no APTAS for the MAXIMUM 3-DIMENSIONAL RECTANGLE PACKING problem with unit cube bin. The same is true also for  $z$ -oriented packings. For  $r$ -packings the same hardness result holds for a bin  $(1, 1, b)$ , where  $b \in (0, \frac{1}{4})$ .*

*Proof.* For oriented packings (i.e., without rotations) we can use the hardness result for 3-SP with the strip  $(1, 1, \infty)$  from Theorem 3: there is a constant  $\rho > 1$  and an infinite family  $\mathcal{F}$  of instances of the 3-SP problem with the strip  $(1, 1, \infty)$  and rectangles with the third side-length equal to 1, such that for some computable function  $\gamma : \mathcal{F} \rightarrow \mathbb{N}$  it is NP-hard to distinguish for  $\mathcal{L} \in \mathcal{F}$  of whether  $\text{OPT}(\mathcal{L}) \leq \gamma(\mathcal{L})$ , or  $\text{OPT}(\mathcal{L}) > \rho \cdot \gamma(\mathcal{L})$ . For any  $\mathcal{L} \in \mathcal{F}$  denote by  $\mathcal{L}'$  rescaled copy of  $\mathcal{L}$  by a factor  $1/\gamma(\mathcal{L})$  in the direction of the  $z$ -axis. Thus after rescaling it is NP-hard for  $\mathcal{L}'$  to decide if  $\text{OPT}(\mathcal{L}') \leq 1$ , or  $\text{OPT}(\mathcal{L}') > \rho$ . In the former case all rectangles of  $\mathcal{L}'$  can be packed into a unit cube bin. In the latter one we easily obtain that less than  $|\mathcal{L}'| - \lfloor (\rho - 1)\gamma(\mathcal{L}) \rfloor$  can be packed into this bin.

For  $z$ -oriented packings we can use the same arguments starting from NP-hard gap of the problem 2-BP<sup>r</sup> with unit bin instead. For  $r$ -packings we scale  $\mathcal{L}$  by a factor  $b/\gamma(\mathcal{L})$  in the  $z$ -direction to reduce the problem to the bin  $(1, 1, b)$ ,  $b \in (0, \frac{1}{4})$ . The special uniform structure of instances in our hardness result for 2-BP<sup>r</sup> imply that all  $r$ -packings for such rescaled instances are, in fact,  $z$ -oriented packings. Thus the results follow as above.  $\square$

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